The Schwinger-Dyson equations: a short introduction to non-perturbative QFT†

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January 6, 2011

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This note contains an overview of the non-perturbative formulation of QED in the Schwinger-Dyson formalism. For the sake of brevity we have chosen to omit most of the technical details and proofs and concentrate on the physical aspects and applications of the SDE. The avid reader is referred to the original work [1] and references therein. The note is organized as follows. In the first section, we formulate the Schwinger-Dyson equations and derive a closed system of integral equations relating the dynamical fermion mass to the renormalization functions of the photon and fermion fields. The second section contains a brief discussion of critical phenomena and the renormalizaiton group. Finally, we present how this formalism can be used to study critical phenomena in QFT. More precisely we study the chiral transition in strong QED in 4 dimensions and discuss a series of phenomena that arise in the supercritical phase such as dynamical mass generation, vacuum stabilization and dimensional transmutation.

†This is a summary of the author's diploma thesis carried out in the Divison of Nuclear Physics and Elementary Particles of the University of Athens under the supervision of Prof. C.N. Ktorides.

1 From perturbative to non-perturbative

The fundamental problem of quantum physics is to determine the time evolution of a system, given its initial states. As we know, a system evolves under the operator e^{-iHt}, so the former problem reduces essentially to the determination of the eigenstates of the Hamiltonian that describes the system. However, except for the case of free fields and some non-realistic interaction models which are of little physical interest, this task is non-trivial. Thus, in order to study a system with interactions one separates the Hamiltonian as follows

$$H = H_0 + H_I, \tag{1}$$

where H_0 is the free Hamiltonian (whose eigenstates are known) and H_I is the part that describes the interactions. What one does in perturbation theory is to express the interactions in terms of free fields as described below. Let $|\Psi(t)\rangle$ be a state that evolves like

$$i\frac{d}{dt}\left|\Psi(t)\right\rangle = \left(H_0 + H_I\right)\left|\Psi(t)\right\rangle . \tag{2}$$

In the absence of interactions the above equation takes the form

$$i\frac{d}{dt}\left|\Psi_0(t)\right\rangle = H_0\left|\Psi_0(t)\right\rangle . \tag{3}$$

Defining the time evolution operators

$$\begin{aligned} \left|\Psi_0(t)\right\rangle &= U_0(t)\left|\Psi_0(-\infty)\right\rangle \equiv U_0(t)\left|i\right\rangle \\ \left|\Psi(t)\right\rangle &= U_0(t)U(t)U_0^\dagger(t)\left|\Psi_0(t)\right\rangle , \end{aligned} \tag{4}$$

one can see that (2) yields

$$i\frac{dU(t,t_0)}{dt} = H_I(t)U(t,t_0), \tag{5}$$

where

$$H_I(t) \equiv U_0^\dagger(t)H_IU_0(t) = \mathrm{e}^{H_0(t-t_0)}H_I\mathrm{e}^{-H_0(t-t_0)} \tag{6}$$

is the interaction Hamiltonian in the interaction picture[1]. Relation (5) is known as the Tomonaga-Schwinger equation. For $t_0 = -\infty$, its solution reads

$$U(t) = 1 + \sum_{n=1}^{\infty}(-i)^n\int_{-\infty}^{t}\ldots\int_{-\infty}^{t_{n-1}} dt_1\ldots dt_n[H_I(t_1)\ldots H_I(t_n)] \tag{7}$$

and using the T-product properties, (7) takes the form

$$U(t) = 1 + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{t} dt_nT\{H_I(t_1)\ldots H_I(t_n)\}. \tag{8}$$

Now in an experiment (e.g. the measurement of a cross section) one needs to define initial and final states for times $t \to -\infty$ and $t \to \infty$ by $|\Psi_{\text{in}}\rangle$, $|\Psi_{\text{out}}\rangle$. These are time-independent and non-interacting asymptotic states of the full Hamiltonian. For a set of initial and final states that are described by the sets of quantum numbers α and β respectively, one can define the S-matrix by

$$S_{\beta\alpha} \equiv \left\langle\Psi_{\text{out}}(\beta)|\Psi_{\text{in}}(\alpha)\right\rangle . \tag{9}$$

[1] $H_I(t)$ is expressed in terms of free fields.

or in terms of asymptotic states [2]

$$\langle \Psi_{\text{out}}(\beta)|\Psi_{\text{in}}(\alpha)\rangle = \lim_{t\to\infty} \langle \beta|U(t)|\alpha\rangle \,. \tag{10}$$

Then (8) gives

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \ldots \int d^4x_1 \ldots d^4x_n T\{\mathcal{H}_I(x_1)\ldots\mathcal{H}_I(x_n)\}. \tag{11}$$

This expression is known as the Dyson expansion of the S-matrix. This serves as the starting point for the formulation of the Feynman rules and the calculation of transition amplitudes that can be further related to physical observables.

1.1 Beyond perturbation theory

Eq. (11) can be symbolically written as

$$S(g) = a_0 + a_1 g + a_2 g^2 + \ldots + a_n g^n + \ldots, \tag{12}$$

where g is the coupling constant and a_i represent the set of Feynman diagrams at the i-th order of perturbation theory. The unprecedented agreement between theory and experiment in QED lies essentially in the fact that g is very small and it thus suffices to keep only the first terms in (12) for the calculation of physical observables. However one can easily see that if $g = \mathcal{O}(1)$ one can no longer neglect the higher order terms and thus perturbation theory is bound to fail with strong coupling[2]. We can see thus that for theories with strong coupling one needs a non-perturbative treatment. Schwinger and Dyson have constructed an infinite system of coupled integral equations [5–7] that relates n-point to $(n+1)$-point correlation functions of the theory providing in principle an exact (i.e. with infinite precision) description of the dynamics. A complete analytical solution of the system of Schwinger-Dyson equations (SDE) remains to date unknown. However the use of truncation schemes and numerical methods can provide access to interesting phenomena that lie in the non-perturbative regime, remaining thus inaccessible by standard perturbation theory.

1.2 The Schwinger-Dyson equations

Schwinger's idea was to introduce an external vector source $J^\mu(x)$ that is coupled to the photon field. The interaction Hamiltonian reads

$$\mathcal{H}_I = -\mathcal{L}_I = e\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) + J^\mu(x)A_\mu(x). \tag{13}$$

The response of the system to this external source (probe) is encoded in the field propagators. One can thus define the (exact) photon propagator as

$$D_{\mu\nu}(x,y) = \left.\frac{\delta\,\langle A_\mu(x)\rangle}{\delta J^\nu(y)}\right|_{J\to 0} = \frac{1}{\langle 0|S|0\rangle} \left.\frac{\delta\,\langle 0|T\{A_\mu(x)S\}|0\rangle}{\delta J^\nu(y)}\right|_{J\to 0}. \tag{14}$$

[2]Even in the case of (weak coupling) QED, (12) is asymptotically divergent [3, 4]. The question if the perturbative expansion is identical to the non-perturbative solution of the theory is known as Borel summability

In a similar way one can define the exact fermion propagator by the relation

$$G(x,y) = \left.\frac{\delta\,\langle\psi(x)\rangle}{\delta\eta(y)}\right|_{\eta\to 0}. \tag{15}$$

For the formulation of the SDE it is also useful to introduce the vertex function

$$\Gamma^{\mu}(x',x'',z) \equiv \frac{1}{e}\frac{\delta G^{-1}(x',x'')}{\delta\hat{\mathcal{A}}_{\mu}(z)}, \tag{16}$$

the vacuum polarization tensor

$$\Pi^{\mu\nu}(x,z) \equiv ie^2 \mathrm{Tr}\left[\gamma^{\nu}\int dx'dx'' G(x,x')\Gamma^{\mu}(x',x'',z)G(x'',x)\right], \tag{17}$$

and the fermion self-energy operator

$$\Sigma^{\mu\nu}(x,x'') \equiv ie^2\gamma^{\nu}\int dz dx' G(x,x')\Gamma^{\mu}(x',x'',z)D(z,x). \tag{18}$$

Their graphic representation is given in the Figure below.

Fig 1. The vacuum polarization tensor (left) and the fermion self-energy operator (right).

It can be proven [1] that the photon and fermion propagators obey the following relations:

$$\begin{aligned}
\left[\Box g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial^{\mu}\partial^{\nu}\right] D_{\mu\rho}(x,y) &= \delta^{\nu}_{\rho}\delta(x-y) + \int dz \Pi^{\mu\nu}(x,z)D_{\mu\rho}(z,y) &(19)\\
\left[i\partial\!\!\!/ - m_0\right] G(x,y) &= \delta(x-y) + \int dx''\Sigma(x,x'')G(x'',y). &(20)
\end{aligned}$$

These are a set of Fredholm integral equations of the second kind. Their solutions are given by the **Schwinger-Dyson equations**

$$\begin{aligned}
D_{\mu\rho}(x,y) &= D^{F}_{\mu\rho}(x-y) + \int dx'dz D^{F}_{\mu\nu}(x-x')\Pi^{\mu\nu}(x',z)D_{\mu\rho}(z,y) &(21)\\
G(x,y) &= S_F(x-y) + \int dx'dx'' S_F(x-x')\Sigma(x',x'')G(x'',y) &(22)
\end{aligned}$$

Their diagrammatic representation is given below.

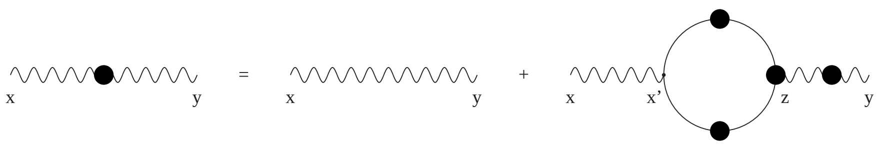

Fig 2. The photon SDE.

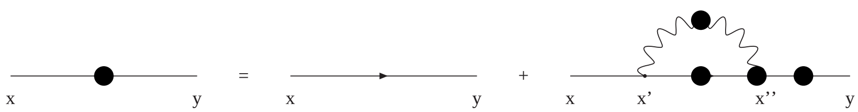

Fig 3. The fermion SDE

The equations (21) and (22) relate the photon and fermion propagators (2-point correlation functions) to the vertex function (3-point correlation function). Since they are coupled, their solution relies on the knowledge of $\Gamma^\mu(x', x'', z)$. However an iterative solution can be written in closed form using the self-energy operators Σ and Π. We remind that the free photon and fermion propagators are given (in momentum space) by

$$
\begin{aligned}
S_F(x-x') &= \frac{1}{(2\pi)^4}\int \frac{\mathrm{e}^{-iq(x-x')}}{\not{p} - m_0 - i\epsilon} dq && (23)\\
D_F^{\mu\nu}(x-y) &= \frac{1}{(2\pi)^4}\int \left[g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\frac{q^\mu q^\nu}{q^2}\right]\frac{-\mathrm{e}^{-iq(x-y)}}{q^2 - i\epsilon} dq. && (24)
\end{aligned}
$$

Taking the Fourier transform of equations (21) and (22) and replacing the free propagators we obtain

$$
\begin{aligned}
D^{\mu\nu}(k) &= D_F^{\mu\nu}(k) + D_F^{\mu\rho}(k)\Pi_{\rho\sigma}(k)D_F^{\sigma\nu}(k) && (25)\\
G(p) &= S_F(p) + S_F(p)\Sigma(p)S_F(p), && (26)
\end{aligned}
$$

which are known as the **Dyson expansion** of the SDE (or Dyson equations). Their diagrammatic representations are given below.

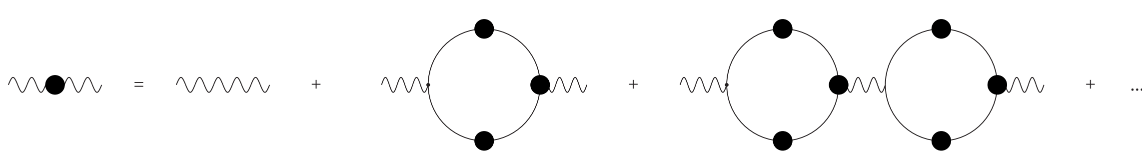

Fig 4. The Dyson expansion of the photon propagator.

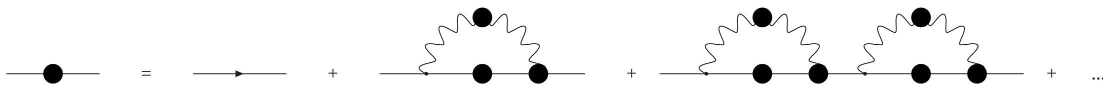

Fig 5. The Dyson expansion of the fermion propagator.

1.3 The $\mathcal{M}, \mathcal{F}, \mathcal{G}$ system

Eq. (26) gives

$$G(p) = \frac{S_F(p)}{1 - S_F(p)\Sigma(p)} = \frac{1}{\not{p} - m_0 - \Sigma(p) - i\epsilon} \tag{27}$$

Comparing (27) with the free fermion propagator, we observe that the result of radiative corrections encoded in the self-energy operator $\Sigma(p)$ is to shift the mass pole from m_0 to $m_0 + \Sigma(p)$. Now remembering that the position of the pole of the free fermion propagator defines the bare fermion mass we are led to the conclusion that $m_0 + \Sigma(p)$ defines the mass of the physical (dressed) fermion, which is a physical observable. We thus define the **mass shift** by

$$\delta m = m - m_0 = \left[\Sigma(\not{p})\right]_{\not{p}=m}. \tag{28}$$

We now expand $\Sigma(\not{p})$ in a Taylor series around $\not{p} = m$. Defining

$$\Sigma_0 \equiv \left[\Sigma(\not{p})\right]_{\not{p}=m} \quad , \quad \Sigma_1 \equiv \left[\frac{\partial \Sigma}{\partial \not{p}}\right]_{\not{p}=m}, \tag{29}$$

we have

$$\Sigma(\not{p}) = \Sigma_0 + (\not{p} - m)\Sigma_1 + \Sigma_r(\not{p}). \tag{30}$$

From (28) we have $\Sigma_0 = \delta m$ and thus

$$G(\not{p}) = \frac{1}{\not{p} - m_0 - \Sigma_0 - (\not{p} - m)\Sigma_1 - \Sigma_r(\not{p}) - i\epsilon} = \frac{1}{(\not{p} - m)(1 - \Sigma_1) - \Sigma_r(\not{p}) - i\epsilon}. \tag{31}$$

Defining now

$$\mathcal{F} \equiv \frac{1}{1 - \Sigma_1}, \tag{32}$$

eq (31) becomes

$$G(\not{p}) = \frac{\mathcal{F}}{\not{p} - m - \mathcal{F}\Sigma_r(\not{p}) - i\epsilon}. \tag{33}$$

Near the pole $\not{p} = m$ the terms $\Sigma_r(\not{p})$ cancel and (33) becomes

$$G(\not{p}) \sim \mathcal{F}\frac{1}{\not{p} - m - i\epsilon}. \tag{34}$$

We observe that the exact fermion propagator differs from the free propagator only by a factor $\mathcal{F}$, known as the **fermion field renormalization constant**. This is equal to the residue of the exact fermion propagator at $\not p = m$. From (34) we observe that the fermion propagator can be written as

$$G(\not p) = \frac{\mathcal{F}(p^2)}{\not p - \mathcal{M}(p^2)} = \frac{\mathcal{F}(p^2)}{p^2 - \mathcal{M}^2(p^2)}\left(\not p + \mathcal{M}(p^2)\right), \tag{35}$$

where

$$\mathcal{M}(p^2) = m_0 + \Sigma(p^2) \tag{36}$$

We observe that in zeroth order perturbation theory (i.e. for non-interacting fermions) (36) gives $\mathcal{M}(p^2) = m_0$, i.e. the fermion mass becomes identical to the bare mass. When interactions between fermions are present, (36) acquires an additional component $\Sigma(p^2)$ called the **dynamical mass** which is due to radiative corrections (fermion self-energy). Eq. (36) is known as the **gap equation**.
Using (35) and the Fourier transform of the exact fermion propagator

$$G^{-1}(p) = \not p - m_0 - \Sigma(p), \tag{37}$$

we get

$$\frac{\not p - \mathcal{M}(p^2)}{\mathcal{F}(p^2)} = \not p - m_0 - \Sigma(p). \tag{38}$$

which leads to

$$\boxed{\frac{\mathcal{M}(p^2)}{\mathcal{F}(p^2)} = m_0 + \frac{1}{4}\mathrm{Tr}\left[\Sigma(p)\right]} \tag{39}$$

and

$$\boxed{\frac{1}{\mathcal{F}(p^2)} = 1 - \frac{1}{4p^2}\mathrm{Tr}\left[\not p\Sigma(p)\right]} \tag{40}$$

We follow the same procedure for the photon field. The vacuum polarization tensor can be written as

$$\Pi^{\mu\nu}(k) = -k^2\left[g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right]\Pi(k^2). \tag{41}$$

Substituting (41) in (25) and using the Ward identity we get[3]

$$D^{\mu\nu}(k) = -\frac{g^{\mu\nu}}{k^2\left[1+\Pi\left(k^2\right)\right]}. \tag{42}$$

Defining

$$\frac{1}{1+\Pi(0)} \equiv \mathcal{G}, \tag{43}$$

we observe that in a scattering procedure, replacing the free photon propagator by the exact one is equivalent to the substitution $e \to \sqrt{\mathcal{G}}e$. In other words, the result of radiative corrections from vacuum polarization is to alter the effective strength of the interaction and the physical charge. In analogy with the mass shift, we can define a **charge shift** by

$$\delta\mathcal{G} = \frac{e^2 - e_0^2}{e_0^2} = \left[\Pi\left(k^2\right)\right]_{k^2=0}. \tag{44}$$

[3]We observe that since $\Pi(k^2)$ is regular at $q^2 = 0$, the exact photon propagator has a pole at $q^2 = 0$, i.e. the photon remains massless in all orders of perurbation theory.

Performing a Taylor expansion of $\Pi(k^2)$ around $k^2 = 0$ and defining

$$\left[\Pi(k^2)\right]_{k^2=0} \equiv \Pi_0, \tag{45}$$

we get

$$\Pi(k^2) = \Pi_0 + \Pi_r(k^2). \tag{46}$$

We observe that $\Pi(0) = \Pi_0$, and thus (43) gives $\mathcal{G} = (1 + \Pi_0)^{-1}$. Substituting in (42), we get

$$D^{\mu\nu}(k^2) = -\frac{g^{\mu\nu}}{k^2\left[1 + \Pi_0 + \Pi_r(k^2)\right] - i\epsilon} = -\mathcal{G}\frac{g^{\mu\nu}}{k^2 + k^2\mathcal{G}\Pi_r(k^2) - i\epsilon}. \tag{47}$$

For real photons, $k^2 \to 0$, and thus the terms $\Pi_r(k^2)$ cancel. Eq. (47) gives then

$$D^{\mu\nu}(k^2) \sim \mathcal{G}\frac{-g^{\mu\nu}}{k^2 - i\epsilon}. \tag{48}$$

We observe that near the mass shell, eq. (48) differs from the free fermion propagator only by a multiplicative factor $\mathcal{G}$, known as the **photon field renormalization constant**. Defining the projection operator

$$\mathcal{P}_{\mu\nu}\Pi^{\mu\nu}(q) = -3q^2\Pi(q^2), \tag{49}$$

it can be proven that the photon SDE can be cast in the form

$$\boxed{\frac{1}{\mathcal{G}(q^2)} = 1 - \frac{iN_f e^2\mathcal{P}_{\mu\nu}}{3q^2(2\pi)^4}\mathrm{Tr}\left[\gamma^\mu \int d^4k G(k)\Gamma^\nu(k,p)G(p)\right]} \tag{50}$$

2 Critical phenomena and the renormalization group

Phase transition is the phenomenon where quantitative changes in the parameters of a system lead to a qualitative change. Phase transitions are usually connected with the breakdown (or restoration) of symmetries. Landau introduced a phenomenological theory for the description of phase transitions based on the notion of an **order parameter** η which is zero in the symmetric phase and non-zero in the asymmetric one. The point where $\eta = 0$ is called **critical point**.
The correlation function $G(\mathbf{r}) = \langle\psi(\mathbf{0})\psi(\mathbf{r})\rangle$ measures the influence of the quantum fluctuations of the field $\psi(\mathbf{0})$ at distance $r = |\mathbf{r}|$. It can be proven [8] that at the critical point

$$\lim_{r\to 0} G_c(\mathbf{r}) \approx \frac{D}{r^{d-2+\eta}}, \tag{51}$$

where d is the dimensionality of spacetime and η here is the so-called **anomalous dimension**. We note that in the classical theory the anomalous dimension vanishes, while in QFT this is true only for massless non-interacting fields. Performing a scale transformation $\mathbf{r} \to \mathbf{r}' = b\mathbf{r}$, (51) becomes

$$G_c(\mathbf{r}) \to G_c'(b\mathbf{r}) = b^{2\omega}\langle\psi(\mathbf{0})\psi(\mathbf{r})\rangle \approx b^{2\omega}\frac{D}{b^{d-2+\eta}r^{d-2+\eta}}, \quad \text{for} \quad r \to \infty. \tag{52}$$

If $\omega = \frac{1}{2}(d - 2 + \eta)$, then (52) is identical to (51). We say that the system exhibits scale invariance or fractal behaviour. Let us now study a system described by the Hamiltonian $\bar{\mathcal{H}}^{(0)} = \bar{\mathcal{H}}^{(0)}[t, h_1, \ldots h_i, \ldots]$, where $(t, h_1, \ldots h_i, \ldots)$ is a set of fields. Following Wilson [9], we suppose that this Hamiltonian defines a subspace in the space of all possible Hamiltonians denoted by $\mathbb{H}$. The renormalization procedure can be described by a transformation $\mathbb{R}_B$ that maps the initial subspace $\bar{\mathcal{H}}^{(0)}$ onto a new subspace $\bar{\mathcal{H}}^{(1)}$, where $(t, h_1, \ldots h_i, \ldots)$ have been renormalized. The set of transformations $\{\mathbb{R}_B\}$ is called **renormalization group**. Writing

$$\mathbb{R}_B[\bar{\mathcal{H}}^{(i)}] = \bar{\mathcal{H}}^{(i+1)}, \quad i = 0, 1, 2, \ldots \tag{53}$$

we can see that under the action of $\mathbb{R}_B$ every point of $\bar{\mathcal{H}}^{(i)}$ (which represents a parameter of the physical system) sweeps out a trajectory in $\mathbb{H}$. The critical point of the initial subspace is mapped onto a new

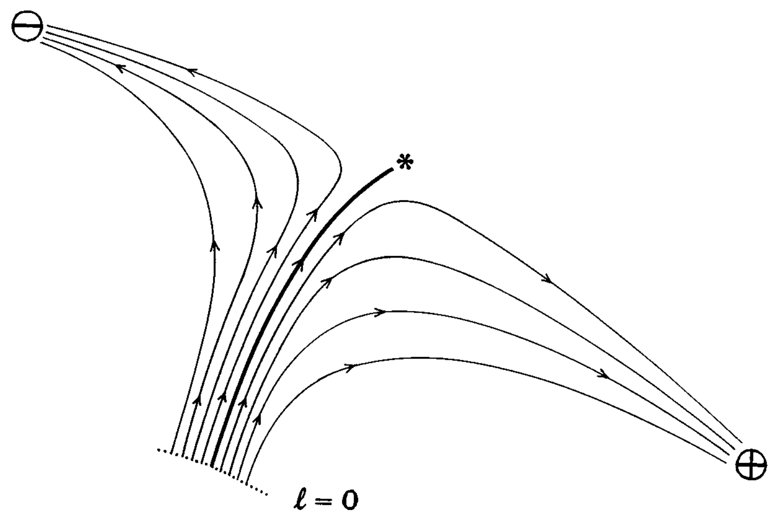

Figure 6: Graphical representation of the renormalization flow in $\mathbb{H}$. The locus $l = 0$ defines the bare Hamiltonians $\bar{\mathcal{H}}^{(0)}$. The thick line corresponds to the critical flow which terminates at the bifurcation point $\star$ (fixed point of the flow). $\oplus$ and $\ominus$ correspond to asymptotic states of high and low temperature respectively. Figure extracted from [10].

critical point and so on, until it reaches a fixed point defined by

$$\mathbb{R}_B[\bar{\mathcal{H}}^{(*)}] = \bar{\mathcal{H}}^{(*)}. \tag{54}$$

As shown in the figure above, the parameters (fields) which lie on the critical flow will remain unchanged under a further application of $\mathbb{R}_B$, while the rest of the parameters will bifurcate to asymptotic states. The system then undergoes a phase transition.

What is important is that systems described by different Hamiltonians can show the same asymptotic behaviour (in other words they have the same critical exponents). This is depicted in the figure below, where $(a), (b), \ldots$ represent subspaces that lie in the domain of attraction of the same fixed point. We say

then that $\bar{\mathcal{H}}^{(*)}$ defines a **universality class**. Fixed points are classified as trivial when they correspond to free fields and non-trivial. One can also classify the different operators according to this scheme. We speak of relevant, irrelevant and marginal operators if their coefficients are increasing, decreasing or constant with the change of the renormalization parameter. We also note that relevant, marginal and irrelevant operators correspond to super-renormalizable, renormalizable and non-renormalizable theories. We observe in the figure above that in the neighborhood of the critical point the system is described only by marginal and relevant operators, while irrelevant operators (which describe the microscopic dynamics

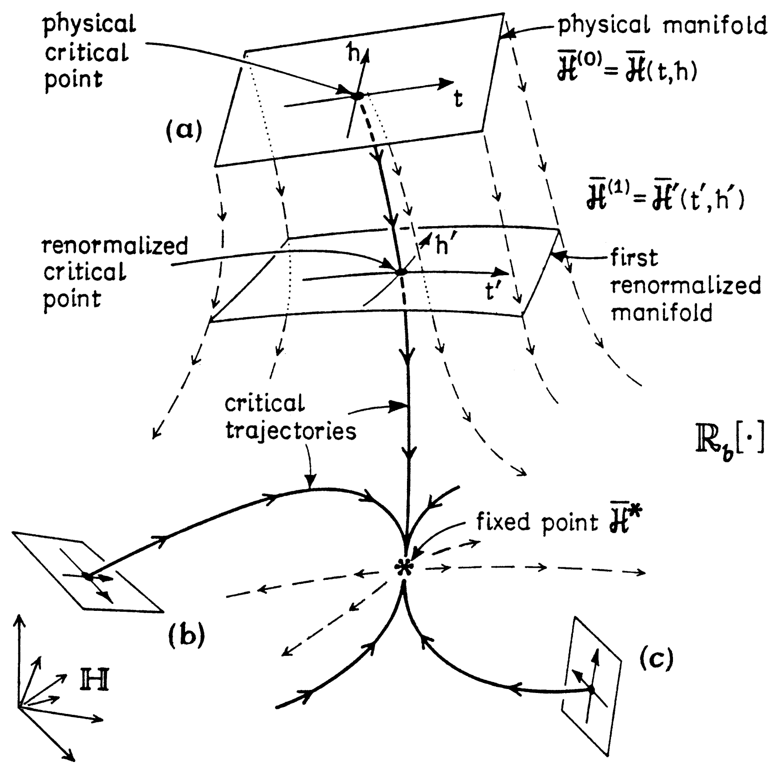

Figure 7: Another representation of $\mathbb{H}$, where $(a),(b),\ldots$ represent the initial manifolds (subspaces $\bar{\mathcal{H}}_i^{(0)}$) corresponding to different physical systems. Dashed lines represent the flows of irrelevant operators while bold lines represent critical flows. $\mathbb{H}$ can in principle contain other fixed points. Figure extracted from [10].

of the system) die out. This explains the appearance of universality and the fact that on macroscopic scales nature seems to be described by renormalizable theories. We will see in the following, the presence of interactions can alter the dimension of certain operators transforming irrelevant operators to marginal or relevant and vice-versa. This phenomenon is called **dimensional transmutation**.

3 Phase transitions in QFT

We saw that the Schwinger-Dyson equations can be formulated in terms of a closed system of integral equations (relations (39),(40),(50)), that will be henceforth referred to as the $\mathcal{MFG}$ system. The $\mathcal{MFG}$ system incorporates the full non-perturbative dynamics of the theory, constituting thus a natural starting point for the study of phenomena that lie beyond the domain of validity of perturbation theory. In this section we will demonstrate how the $\mathcal{MFG}$ system can be used to study phase transitions.
As we saw, the $\mathcal{MFG}$ system comprises an infinite tower of coupled non-linear integral equations and thus an analytical solution in a closed form is impossible. This forces the use of some truncation scheme that reduces the complexity of the system. Truncation schemes consist in neglecting the correlation functions higher than a certain order, thus collapsing the tower to an infinite sum of topologically similar Feyman diagrams. We emphasize that even after the truncation, the system involves an infinite number of Feynman diagrams, hence resting beyond a simple perturbative expansion. Here we employ the simplest

truncation scheme, namely the bare vertex approximation, which consists in setting

$$\Gamma^{\mu}(x', x'', z) = \gamma^{\mu}, \tag{55}$$

in (16), i.e. evaluating only up to 2-point correlation functions. This truncation allows for analytical solutions in the so-called quenched approximation, i.e. the limit where the effect of vacuum polarization is neglected. Formally the quenched approximation consists in setting[4]

$$N_f = 0, \tag{56}$$

in (50), which together with (48) gives

$$D_{\mu\nu}(q) \to D^F_{\mu\nu}(q) = -\frac{1}{q^2}\left[g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right]. \tag{57}$$

The combination of the BVA with the quenched approximation is known as the rainbow approximation, following the characteristic shape of the self-energy operator, given in the Figure below.

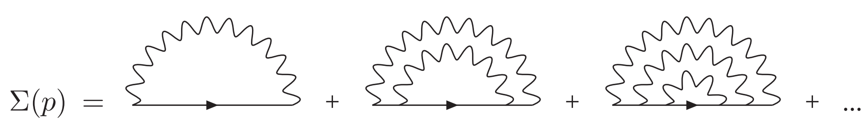

Fig 8. The fermion self-energy operator in the rainbow approximation.

3.1 QED_4 in the rainbow approximation

Substituting the Fourier transform of (18) in the rainbow approximation in (39), we get

$$\begin{aligned}
\frac{\mathcal{M}(p^2)}{\mathcal{F}(p^2)} &= m_0 + \frac{ie^2}{4(2\pi)^4}\int d^4k \mathrm{Tr}\left[\gamma^\mu G(k)\gamma^\nu\right] D^F_{\mu\nu}(k-p) \\
&= m_0 - \frac{ie^2}{4(2\pi)^4}\int d^4k \frac{\mathcal{F}(k^2)}{k^2 - \mathcal{M}^2(k^2)}\mathrm{Tr}\left[\gamma^\mu\left(\not{k} + \mathcal{M}(k^2)\right)\gamma^\nu\right]\frac{1}{q^2}\left[g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right]
\end{aligned} \tag{58}$$

Using (35) and (57) and setting $q = k - p$, we obtain

$$\frac{\mathcal{M}(p^2)}{\mathcal{F}(p^2)} = m_0 - \frac{ie^2}{4(2\pi)^4}\int d^4k \frac{\mathcal{F}(k^2)}{k^2 - \mathcal{M}^2(k^2)}\mathrm{Tr}\left[\gamma^\mu\left(\not{k} + \mathcal{M}(k^2)\right)\gamma^\nu\right]\frac{1}{q^2}\left[g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right]. \tag{59}$$

Calculating the traces and performing a Wick rotation we have

$$\begin{aligned}
\frac{\mathcal{M}(p^2)}{\mathcal{F}(p^2)} &= m_0 - \frac{ie^2}{(2\pi)^4}\int d^4k \frac{\mathcal{F}(k^2)\mathcal{M}(k^2)}{k^2 - \mathcal{M}^2(k^2)}\frac{g^{\mu\nu}}{q^2}\left[g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right] \\
&= m_0 - \frac{ie^2}{(2\pi)^4}(3+\xi)\int d^4k \frac{\mathcal{F}(k^2)\mathcal{M}(k^2)}{k^2 - \mathcal{M}^2(k^2)}\frac{1}{q^2} \\
&= m_0 + \frac{(3+\xi)e^2}{(2\pi)^4}\int d^4k \frac{\mathcal{F}(k^2)\mathcal{M}(k^2)}{k^2 + \mathcal{M}^2(k^2)}\frac{1}{q^2}.
\end{aligned} \tag{60}$$

[4]In other words, in the quenched approximation one regards the fermion fields as "frozen" or non-dynamical.

Switching to spherical coordinates, we obtain after some algebra

$$\frac{\mathcal{M}(p^2)}{\mathcal{F}(p^2)} = m_0 + \frac{a(3+\xi)}{4\pi}\int_0^{\Lambda^2} dk^2 \frac{\mathcal{F}(k^2)\mathcal{M}(k^2)}{k^2+\mathcal{M}^2(k^2)}\left[\frac{k^2}{p^2}\theta(p^2-k^2)+\theta(k^2-p^2)\right], \tag{61}$$

where $a = \frac{e^2}{4\pi}$ and Λ^2 is a momentum cutoff. Starting from (40) and performing a completely analogous calculation, we find

$$\frac{1}{\mathcal{F}(p^2)} = 1 + \frac{a\xi}{4\pi}\int_0^{\Lambda^2} dk^2 \frac{\mathcal{F}(k^2)}{k^2+\mathcal{M}^2(k^2)}\left[\frac{k^4}{p^4}\theta(p^2-k^2)+\theta(k^2-p^2)\right]. \tag{62}$$

3.2 Dynamical mass generation

To continue the calculation we make a particular choice for the gauge, namely the Landau gauge defined by $\xi = 0$. A formal treatment (see [1,11] and references therein) reveals that in the rainbow approximation the Ward identity is satisfied to order $\mathcal{O}\left(\mathcal{M}(q^2)\right)$. This ensures that observable quantities will be gauge independent and thus the results obtained in a particular gauge will be valid for all other gauges. Equations (62) and (50) ($N_f = 0$) give

$$\begin{aligned} \mathcal{F}(p^2) &= 1 \qquad (63)\\ \mathcal{G}(p^2) &= 1. \qquad (64) \end{aligned}$$

The above relations tell us that there should be no running coupling in the rainbow approximation. This was expected from a physical point of view, since the running of the coupling constant is associated with vacuum polarization, which is neglected here by definition. Without a running coupling, one would expect to recover a scale invariant theory. We warn however against this intuition, which will be shown to be false in the following.
We now turn to the calculation of the gap equation which determines the mass of the fermions. As we have seen, the presence of interactions alters the physical mass of the fermions by $\Sigma(p^2)$ (eq. 36). We want to know if this phenomenon can account for the dynamical generation of mass in chiral theories, i.e. if starting from $m_0 = 0$ we can end up with massive fermions, on the sole account of interactions. In the chiral limit, the gap equation (61) becomes

$$\begin{aligned} \mathcal{M}(p^2) &= \frac{3a}{4\pi}\int_0^{\Lambda^2} dk^2 \frac{\mathcal{M}(k^2)}{k^2+\mathcal{M}^2(k^2)}\left[\frac{k^2}{p^2}\theta(p^2-k^2)+\theta(k^2-p^2)\right] \\ &= \frac{3a}{4\pi}\left[\frac{1}{p^2}\int_0^{p^2} dk^2 \frac{k^2\mathcal{M}(k^2)}{k^2+\mathcal{M}^2(k^2)} + \int_{p^2}^{\Lambda^2} dk^2 \frac{\mathcal{M}(k^2)}{k^2+\mathcal{M}^2(k^2)}\right]. \end{aligned} \tag{65}$$

Eq. (65) is a non-linear integral equation of the Hammerstein type. It has been observed [12–16] that such equations arise in the study of phase transitions in a series of different physical systems. In addition phase transitions can be described as bifurcations in a characteristic function that describes the system. Therefore in the above case the existence of a bifurcation point would signal a phase transition. It has been proven [14] that a Hammerstein equation of the form

$$\mathcal{M}(x,\lambda) = \lambda\int_0^{\infty} K(x,y)\frac{y\mathcal{M}(y,\lambda)}{y+\mathcal{M}^2(y,\lambda)}dy. \tag{66}$$

admits a non-trivial solution[5] if an only if $\lambda > \lambda_{\min}$, where $\lambda_{\min}$ is the smallest eigenvalue of the linearized equation

$$\overline{\mathcal{M}}(x,\lambda) = \lambda \int_0^\infty K(x,y)\overline{\mathcal{M}}(y,\lambda)dy. \tag{67}$$

In other words $\lambda_{\min}$ corresponds to the point where a phase transition occurs. In our case λ corresponds to the coupling constant and $\lambda_{\min}$ corresponds to a critical value of the coupling constant a_c where the gap equation acquires a non-trivial solution, i.e. where the dynamical fermion mass becomes non-zero. This phenomenon is known as **dynamical mass generation** and the associated transition, where massless fermions become massive is known as the **chiral transition**. Let us now go back to eq. (65). Linearizing with respect to $\mathcal{M}(k^2)$ and introducing an infrared cutoff[6]

$$\mathcal{M}(p^2) = \frac{3a}{4\pi}\left[\frac{1}{p^2}\int_{\kappa^2}^{p^2} dk^2 \mathcal{M}(k^2) + \int_{p^2}^{\Lambda^2} dk^2 \frac{\mathcal{M}(k^2)}{k^2}\right], \tag{68}$$

which can be converted to the differential equation

$$(p^2)^2 \frac{d^2\mathcal{M}(p^2)}{d(p^2)^2} + 2p^2\frac{d\mathcal{M}(p^2)}{dp^2} + \frac{3a}{4\pi}\mathcal{M}(p^2) = 0, \tag{69}$$

with boundary conditions given by

$$\lim_{p^2\to\kappa^2} \frac{d\mathcal{M}(p^2)}{dp^2} = 0, \tag{70}$$

$$\lim_{p^2\to\Lambda^2}\left[p^2\frac{d\mathcal{M}(p^2)}{dp^2} + \mathcal{M}(p^2)\right] = m_0. \tag{71}$$

The general solution of (69) takes the form

$$\mathcal{M}(p^2) = (p^2)^{-s}. \tag{72}$$

Substituting (72) in (69) we get

$$s^2 - s + \frac{3a}{4\pi} = 0, \tag{73}$$

from where we can easily calculate the value of the critical coupling

$$a_c = \frac{\pi}{3}. \tag{74}$$

Separating the solutions with respect to a_c, we can write[7]

$$\begin{cases} \mathcal{M}(p^2) = C_1\cdot(p^2)^{-\frac{1}{2}-\frac{\sigma}{2}} + C_2\cdot(p^2)^{-\frac{1}{2}+\frac{\sigma}{2}}, & \text{for } a < a_c \\ \mathcal{M}(p^2) = C_1\cdot(p^2)^{-\frac{1}{2}} + C_2\cdot(p^2)^{\frac{1}{2}}, & \text{for } a = a_c \\ \mathcal{M}(p^2) = C_1\cdot(p^2)^{-\frac{1}{2}-\frac{i\tau}{2}} + C_2\cdot(p^2)^{-\frac{1}{2}+\frac{i\tau}{2}}, & \text{for } a > a_c, \end{cases} \tag{75}$$

[5]We note that a trivial solution is one satisfying $\mathcal{M}(p^2) = m_0$ or in the case of chiral fermions $\mathcal{M}(p^2) = 0$.

[6]As we will see this IR cutoff serves to set the scale of the dynamical mass, thus breaking explicitly the scale invariance of the theory.

[7]We remind that a denotes here and in what follows the bare coupling constant.

where $\sigma = \sqrt{1 - \frac{a}{a_c}}$, $\tau = \sqrt{\frac{a}{a_c} - 1}$. Substituting the boundary conditions, one easily sees that for $a \leq a_c$, the only solution is the trivial one $\mathcal{M}(p^2) = 0$, while for $a > a_c$ one gets

$$\begin{array}{rcrcl} C_1(-\frac{1}{2} - \frac{i\tau}{2})\kappa^{-i\tau} & + & C_2(-\frac{1}{2} + \frac{i\tau}{2})\kappa^{i\tau} & = & 0 \\ C_1(\frac{1}{2} - \frac{i\tau}{2})\Lambda^{-i\tau} & + & C_2(\frac{1}{2} + \frac{i\tau}{2})\Lambda^{i\tau} & = & m_0 \end{array} \tag{76}$$

After some lines of algebra we get

$$\frac{\Lambda}{\kappa} = \exp\left(\frac{\pi}{\sqrt{\frac{a}{a_c} - 1}} - 2\right), \tag{77}$$

which is of the form

$$\frac{\Lambda}{m} = \exp\left(\frac{A}{\sqrt{\frac{a}{a_c} - 1}} - B\right). \tag{78}$$

This is known as Miransky's scaling law [17]. The associated beta function that describes the renormalization flow in the supercritical regime is

$$\beta(a) = \frac{\partial a}{\partial \ln \Lambda} = -\frac{2}{3}\left(\frac{a}{a_c} - 1\right)^{3/2}. \tag{79}$$

Let us sum up what we have seen in this section. Starting with chiral fermions and given the fact that vacuum polarization is absent in the rainbow approximation, one would expect that the theory display scale invariance, a fact which should be reflected in the spectrum (e.g. existence of dilaton multiplets). Enlarging the parameter space of the theory to include the bare coupling constant, we see that the coupling strength separates the theory in 2 regimes: a subcritical one, where $a < a_c$ and a supercritical one, where $a > a_c$. In the subcritical phase the gap equation has only a trivial solution $\mathcal{M}(p^2) = 0$, i.e. the fermions remain massless and the theory is scale invariant. The supercritical phase is more subtle. Although $\mathcal{F} = \mathcal{G} = 1$, as in the subcritical phase, relation (78) shows that the supercritical dynamics introduces a new kind of divergence related to to the dynamical mass. This in turn induces a running of the coupling constant, as shown eq. (79). Finally we note that the dynamical mass introduces a scale in the theory and thus explicitly breaks the scale invariance.

3.3 Tachyon condensation: how dynamical mass generation stabilizes the vacuum

Spontaneous chiral symmetry breaking should lead according to the Goldstone theorem to the appearance of massless Nambu-Goldstone bosons. In the case described above the chiral symmetry is broken dynamically, i.e. the order parameter of the theory is the composite operator $\bar{\psi}\psi$. In this case the NG bosons will take the form of chiral condensates, i.e. bound states of fermion-antifermion pairs of opposite chirality. These bound states are described by the Bethe-Salpeter equation, which is a generalization of the SDE for bound states. The BS equation for a pseudoscalar chiral condensate comprised of a fermion a and an antifermion b gives [11]

$$M^{(p)2}_{ab;k} = 2m_a m_b - 32\Lambda^2 \exp\left(\frac{-2k\pi}{\sqrt{\frac{4a}{\pi} - 1}}\right), \quad k = 1, 2, \ldots. \tag{80}$$

We observe that for $m_a = m_b = 0$ and $a > a_c$ (80) implies the existence of tachyon modes with

$$M^{(p)2}_{ab;k} < 0$$

We know however that the mass of a field is related to the effective potential by the relation

$$\frac{d^2 V_{\text{eff}}(\phi_0)}{d\phi^2} = m^2, \quad \text{with } V'_{\text{eff}}(\phi_0) = 0. \tag{81}$$

Thus the appearance of tachyons with $m^2 < 0$ denotes that the field is at a local maximum of the effective potential, i.e. the vacuum of the theory is unstable. We observe thus that in the supercritical regime, the chirally symmetric state is unstable. For $m_a = m_b = m_{dyn}$ and $a > a_c$, chiral symmetry is dynamically broken and by the virtue of the Goldstone theorem there appear N^2 massless pseudoscalar bosons obeying the relation

$$M^{(p)2}_{ab;k} = 0 \Rightarrow m^{(k)2}_{dyn} = \frac{1}{2}\Lambda^2 f^{(k)}(a), \tag{82}$$

with

$$f^{(k)}(a) = 32 \exp\left(\frac{-2k\pi}{\sqrt{\frac{4a}{\pi} - 1}}\right) \qquad k = 1, 2, \ldots. \tag{83}$$

Equation (80) shows that the mass of the condensate $M^{(p)2}_{ab}$ grows with the mass of its constituents m_a, m_b. At the value

$$m^2 = m^2_{dyn} = \frac{1}{2}\Lambda^2 f^{(1)}(a), \tag{84}$$

the pseudoscalar tachyons are converted to massless pseudoscalar NG bosons. The situation for scalar fermion-antifermion pairs is similar. In the supercritical regime, massless scalar condensates are tachyonic, while after the chiral symmetry breaking they acquire a positive mass. This phenomenon is depicted in the Figure 9. The analysis of this sections shows the following. In the supercritical regime $a > a_c$, dynamical breaking of the chiral symmetry $U_L(N) \times U_R(N)$ leads to the appearance of $2N^2$ chiral condensates (N^2 of which are pseudoscalar and N^2 are scalar). In the symmetric phase ($m = 0$) these condensates are tachyonic. The breakdown of the chiral symmetry generates a dynamical mass for the fermions and thus the N^2 pseudoscalar tachyons are converted into massless NG bosons and the rest N^2 scalar tachyons aquire a positive mass. It is the dynamical mass generation which stabilizes the vacuum. On a more formal level, this process can be viewed as the evolution of the initial tachyonic modes, which are are described as we saw by local maxima of the effective potential. Symmetry breaking consists in these modes acquiring a vacuum expectation value and reaching the minimum of the effective potential. By analogy with similar processes occurring in solid-state physics (e.g. Bose-Einstein condensation) this process is sometimes referred to as tachyon condensation. In the case of (quenched) QED, the chiral symmetry group is $U_L(1) \times U_R(1)$. We thus expect that in the supercritical phase there appears 1 massless pseudoscalar NG boson (parapositronium) and 1 massive scalar boson (orthopositronium).

3.4 Dimensional transmutation

Scale transformations are associated with a conserved current called dilatation current, with the corresponding generator given by

$$\mathbb{D} = \int d^3x \mathbb{D}_0(x) \tag{85}$$

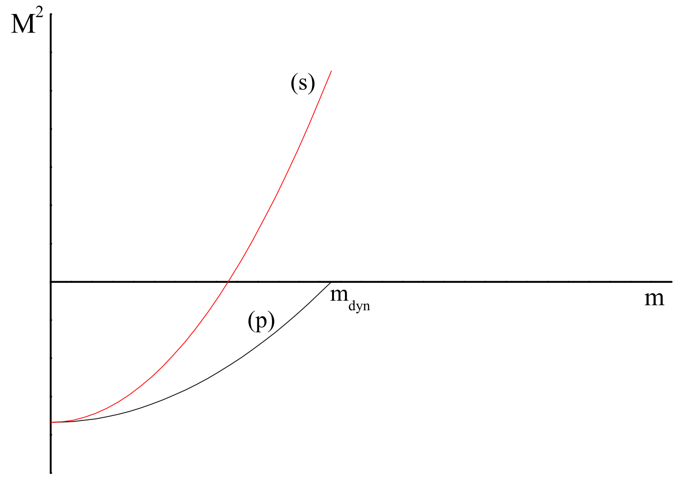

Figure 9: *The mass of pseudoscalar (p) and scalar (s) chiral condensates as a function of the fermion mass m.*

The conserved quantum number d_ϕ corresponding to this generator is called dynamical dimension and is given by

$$-i\,[\mathbb{D}, \phi(x)] = (d_\phi + x^\mu \partial_\mu)\, \phi(x). \tag{86}$$

Massless non-interacting fields are invariant under scale transformations. In this case the dynamical dimension is equal to the canonical or engineering dimension $d_{c;\phi}$ of the field ϕ. Free bosonic fields in d-dimensions have a canonical dimension of $\frac{1}{2}(d-2)$, while fermionic fields have a canonical dimension of $\frac{1}{2}(d-1)$. Thus for the case of QED, we have

$$\begin{cases} [\psi] = [\bar{\psi}] & = & \frac{3}{2}, \\ [A_\mu] & = & 1 \end{cases} \tag{87}$$

and the QED Lagrangian contains only marginal operators. We also note that 4-fermion operators of the kind $(\bar{\psi}\psi)^2$ are irrelevant or non-renormalizable, since

$$[(\bar{\psi}\psi)^2] = 6 > d. \tag{88}$$

In the presence of interactions, the fields acquire an anomalous dimension

$$\gamma_\phi = |d_{c;\phi} - d_\phi|\,. \tag{89}$$

We see thus that a large anomalous dimension can convert irrelevant operators to marginal or relevant. We examine this phenomenon in the supercritical regime of quenched QED$_4$. It can be proven [11] that the anomalous dimension for the operators $\bar{\psi}\lambda^a\psi$ and $\bar{\psi}\gamma_5\lambda^a\psi$ is given by

$$\gamma_m = -\frac{\partial \ln Z_m^{(\rho)}}{\partial \ln \Lambda}, \tag{90}$$

with

$$Z_m^{(\rho)} = \begin{cases} \left(\frac{\Lambda^2}{\rho^2}\right)^{\frac{\sigma-1}{2}} & , \qquad a < a_c \\ \frac{\rho}{\Lambda} & , \qquad a \geq a_c. \end{cases} \tag{91}$$

Eq. (90) gives by virtue of (91)

$$\gamma_m = \begin{cases} 1-\sigma & , \qquad a < a_c \\ 1 & , \qquad a \geq a_c, \end{cases} \tag{92}$$

where $\sigma = \sqrt{1-\frac{a}{a_c}}$. In the subcritical regime, we have $\sigma \simeq 1$ and thus $\gamma_m \simeq 0$. Then by (89) we observe that only operators with a canonical dimension $d_c = 4$ are renormalizable. However, in the supercritical regime, eq. (92) shows that the composite operators $\bar{\psi}\lambda^a\psi$ and $\bar{\psi}\gamma_5\lambda^a\psi$ acquire a large anomalous dimension and thus the 4-fermion operator

$$\frac{1}{\Lambda^2}\sum_a \left[\left(\bar{\psi}\lambda^a\psi\right)^2 + \left(i\bar{\psi}\gamma_5\lambda^a\psi\right)^2\right], \tag{93}$$

becomes renormalizable in 4-dimensions [18]. This demonstrates that strong QED$_4$ is a non-complete theory and 4-fermion operators of the kind (93) have to be added to the QED Lagrangian, in order to have a complete description of the dynamics. A model which incorporates QED with 4-fermion interactions is the so-called gauged Nambu-Jona-Lasinio (gNJL) model, whose interaction Lagrangian is given by [19]

$$\mathcal{L}_I = e\bar{\psi}^a\gamma^\mu\psi^a A_\mu + \frac{G}{2}\left[\left(\bar{\psi}^a\psi^a\right)^2 + \left(i\bar{\psi}^a\gamma_5\psi^a\right)^2\right], \tag{94}$$

where G denotes the (bare) coupling constant of 4-fermion interactions.

3.5 Summary: The phase diagram

Since the mathematical structure of QFT is based on local operators, a cutoff scale Λ is necessary to obtain well-defined equations. A natural question which arises then is the behavior of the theory in the infrared and ultraviolet limit defined by $\Lambda \to 0$ and $\Lambda \to \infty$ respectively. The limiting behavior of the theory is described by the fixed points of the beta function of the renormalization group. For instance, in non-abelian field theories which are asymptotically free, the coupling constant becomes smaller as the energy increases, hence $\beta_0 = 0$ constitutes an infrared fixed point. In abelian theories on the other hand, the coupling grows with energy and the existence of the continuum (or local) limit $\Lambda \to \infty$ is a completely non-perturbative problem[8].

The Landau pole

The QED running coupling constant was calculated by Landau, Pomeranchuk and Fradkin in the 1-loop approximation and was found to be [20, 21]

$$\frac{1}{e_R^2} - \frac{1}{e^2} = \frac{N_f}{6\pi^2}\ln\frac{\Lambda}{m_R}, \tag{95}$$

where e_R and e denote the renormalized and bare charge respectively and m_R is the renormalized mass. Keeping e constant, eq. (95) implies

$$e_R \stackrel{\Lambda\to\infty}{\longrightarrow} 0. \tag{96}$$

[8]One encounters an analogous situation in the study of the infrared limit of non-abelian theories

This is known as the zero charge situation and signals the absence of interactions in the local limit. In other words the local limit of the theory is trivial. Keeping e_R constant, (95) gives

$$e \overset{\Lambda\to\Lambda_L}{\longrightarrow} \infty, \tag{97}$$

with

$$\Lambda_L = m_R \exp\left[\frac{6\pi^2}{N_f e_R^2}\right]. \tag{98}$$

The divergence of the bare charge at the energy scale Λ_L is called Landau pole. This phenomenon arises in all non asymptotically free theories and thus implies that abelian field theories are trivial. We warn however that there is a potential loophole in the above. The above result was obtained by perturbation theory, while, as we have already mentioned, the ultraviolet regime of QED is non-perturbative. Thus the appearance of this divergence might be an artifact of perturbation theory. Gell-Mann and Low have demonstrated that the existence of a non-trivial local limit is possible only if there is a UV stable fixed point in the renormalization group flow [22]. We will demonstrate that this is indeed the case with strong QED$_4$ in the quenched approximation. We note for completeness that in the Standard Model $\Lambda_L \simeq 10^{34}$ GeV while in the MSSM with 2 Higgs doublets $\Lambda_L \simeq 10^{17}$ GeV which lies below the Planck scale $\Lambda_{Pl} \simeq 10^{19}$ GeV [23].

The UV fixed point

Solving (78) with respect to a, we obtain

$$a = a_c + \frac{A^2 a_c}{\ln^2\left[\frac{\Lambda}{m_{dyn}} \mathrm{e}^B\right]} \overset{\Lambda\to\infty}{\longrightarrow} a_c. \tag{99}$$

We see thus that as the energy increases the coupling constant tends to a constant value a_c. Comparing also with eq. (79) we see that a_c is a UV stable fixed point of the renormalization flow. From the viewpoint of the theory of phase transitions, a_c signals a phase change, separating the chirally symmetric $a < a_c$ phase from the chirally asymmetric one $a > a_c$. Moreover this phase change is of 2nd order, since the order parameter

$$m_{dyn} = \Lambda \exp\left[-\frac{A}{\tau} + B\right] \tag{100}$$

tends continuously to 0 as $a \to a_c^+$. This fixed point defines a universality class that is not the same as subcritical (quenched) QED. As mentioned above, in this universality class, composite chiral fields $\pi^a \sim \bar{\psi}\gamma_5\lambda^a\psi$ and $\sigma^a \sim \bar{\psi}\lambda^a\psi$ arise as new degrees of freedom.

The gNJL model

Starting from (94), it can be proven that the gap equation of the gNJL model in the rainbow approximation takes the form [11]

$$\begin{aligned}\mathcal{M}(p^2) &= \frac{3a}{4\pi}\int_0^{\Lambda^2} dk^2 \frac{k^2\mathcal{M}(k^2)}{k^2+\mathcal{M}^2(k^2)}\left[\frac{\theta(p^2-k^2)}{p^2} + \frac{\theta(k^2-p^2)}{k^2}\right] \\ &+ \frac{\kappa}{\Lambda^2}\int_0^{\Lambda^2} dk^2 \frac{k^2\mathcal{M}(k^2)}{k^2+\mathcal{M}^2(k^2)},\end{aligned} \tag{101}$$

where $\kappa = \frac{N_f G \Lambda^2}{4\pi^2}$. Eq. (101) can be converted to the differential equation

$$\frac{d}{dp^2}\left[p^2 \frac{d\mathcal{M}(p^2)}{dp^2}\right] + \frac{3a}{4\pi}\frac{\mathcal{M}(p^2)}{p^2 + \mathcal{M}(p^2)} = 0, \tag{102}$$

subject to the boundary conditions

$$\lim_{p^2 \to 0}\left[\left(p^2\right)^2 \frac{d\mathcal{M}(p^2)}{dp^2}\right] = 0, \tag{103}$$

$$\lim_{p^2 \to \Lambda^2}\left[\left(1 + \frac{4\pi\kappa}{3a}\right) p^2 \frac{d\mathcal{M}(p^2)}{dp^2} + \mathcal{M}(p^2)\right] = 0. \tag{104}$$

Linearizing (102) we find that in the local limit $\Lambda \to \infty$ the non-trivial solutions satisfy

$$\kappa_c(a) = \frac{1}{4}(1-\sigma)^2, \quad \text{for } a \leq a_c, \tag{105}$$

where $\sigma = \sqrt{1 - \frac{a}{a_c}}$. Eq. (105) defines a line of critical points in the plane (a, κ_c), depicted in the figure below. It has also been proven that the gNJL model and quenched QED_4 belong in the same universality

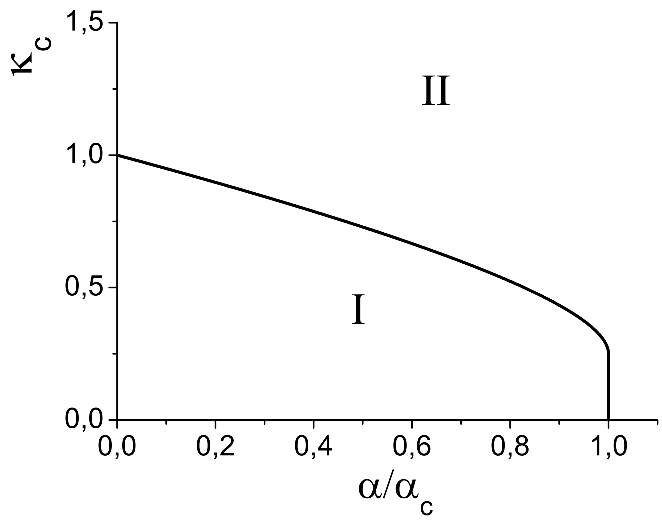

Figure 10: *The critical line of the gNJL model in the rainbow approximation. Regions I and II correspond to the chirally symmetric and asymmetric phases respectively.*

class described by the critical exponents [24]

$$a = \frac{2(\sigma-1)}{\sigma}, \quad \beta = \frac{2-\sigma}{2\sigma}, \quad \gamma = 1, \quad \delta = \frac{2+\sigma}{2-\sigma}, \quad \nu = \frac{1}{2\sigma}, \quad \eta = 2(1-\sigma). \tag{106}$$

We observe from Fig. 3.5 that the point $(1,0)$ defines the critical point of the pure NJL model, whose critical exponents are calculated by the relations (106) with $\sigma \to 1$ and are found to be equal to mean field critical exponents

$$a_{cl} = 0, \quad \beta_{cl} = \frac{1}{2}, \quad \gamma_{cl} = 1, \quad \delta_{cl} = 3, \quad \nu_{cl} = \frac{1}{2}, \quad \eta_{cl} = 0. \tag{107}$$

The point $(0, 1)$ defines the pure QED limit. This point constitutes the UV stable fixed point found above, where the theory describes composite fields $\pi^a \sim \bar{\psi}\gamma_5\lambda^a\psi$ and $\sigma^a \sim \bar{\psi}\lambda^a\psi$ interacting with Yukawa type forces. The critical line (105) interpolates thus between the critical points of QED and the NJL model in the rainbow approximation.

The quenched QED phase diagram

Let us sum up the above discussion for the phase diagram of QED_4 in the rainbow approximation.

<u>Subcritical phase $a < a_c$</u>

In the subcritical phase, for $N_f = 0$ there is no running of the coupling constant, thus

$$\beta(a) = 0 \tag{108}$$

for every $a < a_c$. In other words, every value of the bare coupling constant a defines a trivial IR stable fixed point and the theory is scale invariant.

<u>Supercritical phase $a > a_c$</u>

In the supercritical phase, dynamical chiral symmetry breaking leads to a new divergence associated with the dynamical mass. The renormalization of the dynamical mass induces in turn a running coupling according to eq. (99). The renormalization flow is described by the beta function (79), which in the local limit $\Lambda \to \infty$ tends to the UV stable fixed point $a = a_c$. Hence the theory is non-trivial in the supercritical regime. The chiral transition is of 2nd order and the fixed point a_c defines a different universality class from perturbative QED which is that of the gNJL model describing composite chiral fields $\pi^a \sim \bar{\psi}\gamma_5\lambda^a\psi$ and $\sigma^a \sim \bar{\psi}\lambda^a\psi$ interacting with Yukawa type forces.

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